New Type of Second Order Tetrahedral Edge Element by Reducing Edge Variables for Quasi-static Field Analysis

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In this paper, we propose a new type of second order tetrahedral edge element for the quasi-static electromagnetic field analysis. The unification of variables of two edges on each side of the conventional element provides for the elimination of one variable on the side, with the system matrix of the finite elements remaining singular. By using the proposed type of element in the quasi-static analysis of a simple model, it is demonstrated that the element is superior to the conventional second order element in terms of computational time.

***Index Terms*— Quasi-static field, second order tetrahedral edge element, unification of variables.**

I. Introduction

The use of edge elements can give fast and accurate electromagnetic field computation. In order to obtain accurate solutions, various kinds of high order elements are proposed [1]-[5]. However, such elements bring an increase in unknown variables and larger computational costs.

It is well known that the elimination of variables based on tree-gauging causes deterioration of the convergence rate of ICCG method [6]. When the number of eliminated variables is less than the number of tree edges, known as partial-gauging, the system matrix remains singular. Nevertheless, the number of ICCG iterations increases compared with ungauged computation [6].

In order to address this issue, the second order element with partial tree-gauging is discussed [7]. Additionally, we propose a new type of hexahedral second order element for magnetostatic field analysis [8]. The element has an identical variable of two edges on each side and the shape functions are approximated to be constant along the side. The element enables the remarkable reduction of computational costs as well as the number of unknown variables.

In this paper, we extend the hexahedral element to a tetrahedral element and apply it to quasi-static problems.

II. Finite Element System Matrix

If we assume that all fields are sinusoidal functions of time, the governing equations using the magnetic vector potential $\boldsymbol{A}$ and the electric scalar potential ϕ for quasi-static electromagnetic fields become

$$\nabla\times(\nu\nabla\times\boldsymbol{A})+\mathrm{j}\omega\sigma(\boldsymbol{A}+\nabla\phi)=\boldsymbol{J}_0, \tag{1}$$

$$\mathrm{j}\omega\nabla\cdot\{\sigma(\boldsymbol{A}+\nabla\phi)\}=0, \tag{2}$$

where $\omega = 2\pi f$, ν, σ, $\boldsymbol{J}_0$ and f are the angular frequency, the magnetic reluctivity, the electric conductivity, the external current density and the frequency, respectively. The potential ϕ satisfies the electric field intensity $\boldsymbol{E} = -\mathrm{j}\omega(\boldsymbol{A} + \nabla\phi)$ to symmetrize the finite element system matrix. In the non-conductive region the electric conductivity is set to be zero and (2) is ignored.

Let $\boldsymbol{x}$ and $\boldsymbol{b}$ denote the unknown vector and right-hand side vector. Due to the weighted residual method, we get the weak form

$$K\boldsymbol{x}=\boldsymbol{b}, \tag{3}$$

where K is the finite element system matrix and $\boldsymbol{x}$ is the solution vector consisting of unknown $\boldsymbol{A}$ and ϕ.

III. Edge Element

In edge element, $\boldsymbol{A}$ and ϕ can be approximated by edge and nodal shape functions $\boldsymbol{N}_{IJ}$, N_I as

$$\boldsymbol{A}=\sum_{IJ}\boldsymbol{N}_{IJ}A_{IJ}, \tag{4}$$

$$\phi=\sum_{I}N_I\phi_I, \tag{5}$$

where I and J the nodes indices, IJ denotes the direction of the edge from node I to node J. $\boldsymbol{N}_{IJ}$ and N_I have following properties:

$$\int_{e_{KL}}\boldsymbol{N}_{IJ}\cdot\mathrm{d}\boldsymbol{s}=\delta_{(IJ)(KL)}, \tag{6}$$

$$N_I\big|_{at\ node\ J}=\delta_{IJ}. \tag{7}$$

Here, d$\boldsymbol{s}$ and $\delta_{(IJ)(KL)}$ denote the line element vector along edge e_{KL} and the Kronecker delta, respectively.

In this paper, a geometrical edge of a finite element is called *side*, whereas the location related to a vector variable is called *edge*. In the case of second order elements, sides and edges are not the same.

A. Conventional Second Order Edge Element

Fig. 1(a) shows the second order tetrahedral element we suggested previously [3][4]. The edge shape functions $\boldsymbol{N}$ and nodal functions N of the element can be described as follows:

$$\boldsymbol{N}_{il}=(4\lambda_i-1)(\lambda_i\nabla\lambda_j-\lambda_j\nabla\lambda_i), \tag{8}$$

$$\boldsymbol{N}_{nm}=4\lambda_k(\lambda_i\nabla\lambda_j-\lambda_j\nabla\lambda_i), \tag{9}$$

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$$N_i = (2\lambda_i - 1)\lambda_i, \tag{10}$$

$$N_l = 4\lambda_i\lambda_j, \tag{11}$$

where i, j and k are nodes at the vertices, and l, m and n are mid-nodes on the edges i-j, j-k, k-i, respectively. $\boldsymbol{N}_{il}$ and $\boldsymbol{N}_{nm}$ are the functions of edges (black arrow vectors in Fig. 1(a)) on sides and the ones of edges (gray arrow vectors) on facets, respectively.

Three edges can be defined on each facet in the element originally. Since the number of independent edges on the facet is two, one edge can be eliminated. It offers the *local* tree-gauging on the facet.

B. New Type of Second Order Edge Element

We consider a particular side of a conventional second order element shown in Fig. 1(b). Fig. 2(a) shows edge 1 and edge 2 of a side with their corresponding shape functions $\boldsymbol{N}_1$ and $\boldsymbol{N}_2$, respectively.

Let A_1 and A_2 denote the potentials on edges 1 and 2, assumed to be the same value A_{unif} / 2. The magnetic vector potential $\boldsymbol{A}$ is rewritten as

$$\begin{aligned} \boldsymbol{A} &= \boldsymbol{N}_1 A_1 + \boldsymbol{N}_2 A_2 \\ &= (\boldsymbol{N}_1 + \boldsymbol{N}_2) A_{unif} / 2 \\ &= \boldsymbol{N}_{unif} A_{unif}, \end{aligned} \tag{12}$$

where $\boldsymbol{N}_{unif}$ as shown in Fig. 2(b), is a new shape function on the side and satisfies (13).

$$\boldsymbol{N}_{unif} = (\boldsymbol{N}_1 + \boldsymbol{N}_2)/2. \tag{13}$$

A_{unif} is the line integral of vector potential on the side, with two edges unified into one. Therefore, the shape functions $\boldsymbol{N}$ of edges (solid arrow vectors in Fig. 1(b)) on the sides are

$$\boldsymbol{N}_{ij} = \{2(\lambda_i + \lambda_j) - 1\}(\lambda_i \nabla\lambda_j - \lambda_j \nabla\lambda_i), \tag{14}$$

instead of (8). The shape functions $\boldsymbol{N}$ of edges (dotted arrow vectors in Fig. 1(b)) on the facets are not changed.

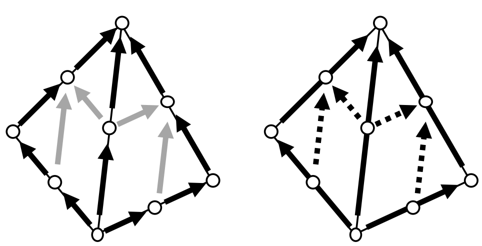

Fig. 1. Second order tetrahedral elements. (a) Conventional (b) Proposed.

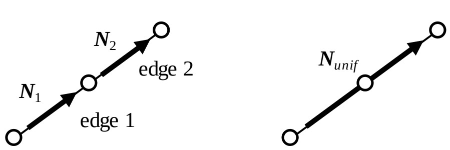

Fig. 2. Unification of two edges. (a) Two edges, (b) Unified edge.

IV. Numerical Results

In order to demonstrate the effectiveness of the proposed element, a simple model shown in Fig. 3 is analyzed. $\boldsymbol{B}$, $\boldsymbol{H}$, $\boldsymbol{n}$ in Fig. 3 are the flux density, the magnetic field intensity and normal unit vector on the boundaries, respectively. The model is a cube region with applied uniform 50 Hz field (B_z) in the z-direction. $\boldsymbol{H}_t$ represents the boundary condition to apply the field. A non-magnetic conductive body (material: Cu) is placed at the center of the region. The conductivity of the body is 5.82×10^{7}S/m. One-eighth of whole region is analyzed due to symmetry.

Let γ_{n} be $n_{\mathrm{n}}/n_{\mathrm{c}}$ where n_{n} and n_{c} are the number of ICCG iterations with the proposed and conventional elements. γ_{c} represents the ratio of computational time with proposed element to that of the conventional ones. γ_{n} is almost one as shown in Fig. 4, the convergence of ICCG method is not slow. As a result, the proposed type of element reduces the computational time.

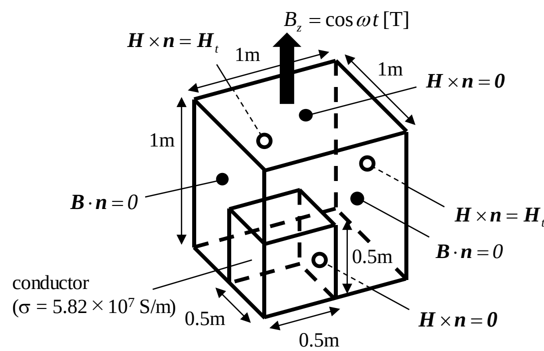

Fig. 3. Uniform field model.

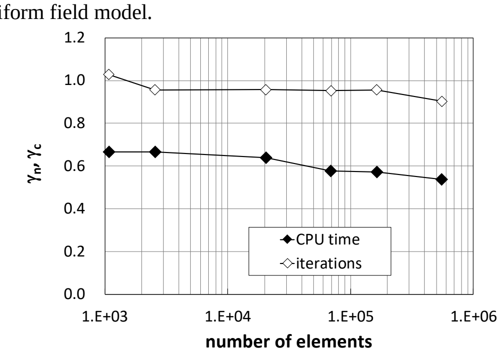

Fig. 4. Numerical results.

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